COXETER COVERS OF THE CLASSICAL COXETER GROUPS

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ABSTRACT. Let $C(T)$ be a generalized Coxeter group, which has a natural map onto one of the classical Coxeter groups, either B_n or D_n. Let $C_Y(T)$ be a natural quotient of $C(T)$, and if $C(T)$ is simply-laced (which means all the relations between the generators has order 2 or 3), $C_Y(T)$ is a generalized Coxeter group, too . Let $A_{t,n}$ be a group which contains t Abelian groups generated by n elements. The main result in this paper is that $C_Y(T)$ is isomorphic to $A_{t,n} \rtimes B_n$ or $A_{t,n} \rtimes D_n$, depends on whether the signed graph T contains loops or not, or in other words C(T) is simply-laced or not, and t is the number of the cycles in T. This result extends the results of Rowen, Teicher and Vishne to generalized Coxeter groups which have a natural map onto one of the classical Coxeter groups.

1. Introduction

Coxeter Groups is an important class of groups which is used in the study of symmetries, classifications of Lie Algebras and in other subjects of Mathematics.

In [5], there is a description of Coxeter groups from which there is a natural map onto a symmetric group. Such Coxeter groups have natural quotient groups related to presentations of the symmetric group on an arbitrary set T of transpositions.

These quotients, which are denoted by $C_Y(T)$, are a special type of the generalized Coxeter groups defined in [1] by a signed Coxeter diagram, where in addition to the regular Coxeter relations, which arise from the graph, every signed cycle, where the multiplication of the signs are negative, admits an extra relation. $C_Y(T)$ is a class of groups where every negatively signed cycle is a triangle. Hence, every extra relation has a form: $(x_1x_2x_3x_2)^2 = 1$, where x_1, x_2 and x_3 are the vertices of the negatively signed triangle.

The group $C_Y(T)$ also arises in the computation of certain invariants of surfaces (see [6]).

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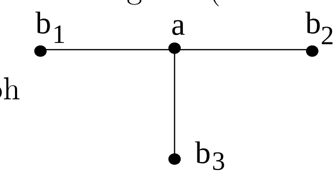
The paper [5] deals with the class of Coxeter groups, whose Coxeter diagram (the dual diagram to the diagram introduced in [5]) does not have a subgraph 

(a, b_1, b_2 and b_3 are Coxeter generators, $(ab_1)^3 = (ab_2)^3 = (ab_3)^3 = 1$ and $(b_1b_2)^2 = (b_1b_3)^2 = (b_2b_3)^2 = 1$) (see [5, Remark 7.13]).

This paper extends the results of [5] for a wider class of Coxeter groups $C(T)$, and $C(T)$ can be also sometimes a generalized Coxeter group [1] where the natural homomorphism is onto one of the classical Coxeter groups A_n, B_n, D_n (which have of course a homomorphism onto S_n). But there are still Coxeter groups $C(T)$ (even simply-laced) which do not have any homomorphism onto any of the classical Coxeter groups, for example, $C(T)$ can not be anyone of the exceptional Coxeter groups. In case of the configuration which mentioned above (which is allowed in our case), two among three vertices (b_i and b_j) satisfy $m_{(b_i,x)} = m_{(b_j,x)}$, for every Coxeter generator x, where $m_{(b_i,x)}$ denotes the order of b_ix in $C(T)$ (the regular notation in Coxeter groups).

Let us briefly recall the definitions and properties of the groups A_n, B_n, D_n and the exceptional Coxeter groups (see [3, page 32]). It is well known [3], that $B_n \cong \mathbb{Z}_2 \wr S_n$ (wreath product) and D_n is a subgroup of B_n of index 2. One can present B_n and D_n as groups of signed permutations, and then present graphs of B_n and D_n as follows: The edges in the

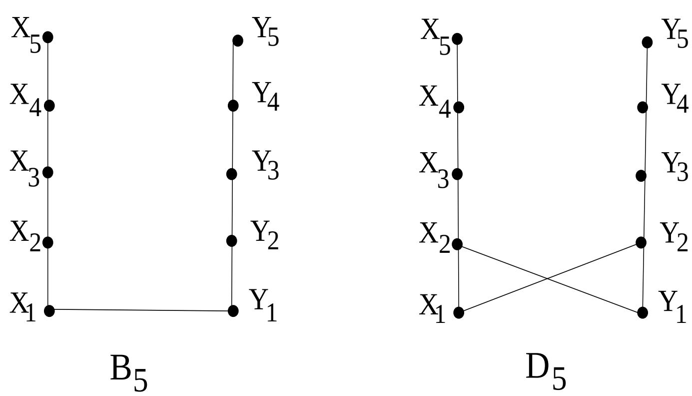


Figure 1.

graph which corresponds to B_n are

$$s_0 = (x_1, y_1), s_1 = (x_1, x_2)(y_1, y_2), s_2 = (x_2, x_3)(y_2, y_3),$$

$$s_3 = (x_3, x_4)(y_3, y_4), s_4 = (x_4, x_5)(y_4, y_5),$$

and the edges in the graph which correspond to D_n are

$$s_{\bar{1}} = (x_1, y_2)(x_2, y_1), s_1 = (x_1, x_2)(y_1, y_2), s_2 = (x_2, x_3)(y_2, y_3),$$
$$s_3 = (x_3, x_4)(y_3, y_4), s_4 = (x_4, x_5)(y_4, y_5).$$

We note that all the generators of D_n are presented by a pair of edges. The generators of B_n, apart from s_0, are presented by pairs of edges, too. This form is analogical to the $2n$ permutation presentation of B_n and D_n, where s_i are presented by a product of two transpositions (s_0 is presented by a single transposition in B_n).

In Section 2 we define the group $C(T)$ which has a natural map onto one of the classical Coxeter groups. A diagram for $C(T)$ (e.g. Figure 2) is analogical to the diagram which was introduced in [5], while in our case, most of generators are presented by a couple of edges, and only specific generators are presented by a single edge.

In Section 3 we introduce a much more convenient presentation of $C(T)$ by reduced diagrams. These diagrams are signed graphs (see [1]), where the edges of the graph are signed either by 1 or -1. Signed graphs are subject to a relation of the form $(u_1 \cdot u_2 \cdots u_{n-1} u_n u_{n-1} \cdots u_2)^2 = 1$ for every cycle with odd number of sign -1 (similarly to [1], which we call anti-cycle. Note that this type of relations appears in [1], but in a dual form, where the generators are vertices and not edges. Due to this additional relation which arises from an anti-cycle, there are signed graphs T, where $C(T)$ is a generalized Coxeter group (Coxeter group with additional relations, which arise from negatively signed cycles, or anti-cycles). We assume that $C(T)$ is connected signed graph, and $C(T)$ does contain a loop or at least one anti-cycle (Otherwise the theorem is isomorphic to the Theorem in [5]).

In Section 4 we classify the relations which arise in the quotient $C_Y(T)$ of $C(T)$. In addition to the anti-cyclic relation, there are other three types of relations which arise in $C_Y(T)$.

In Section 5 we classify the cyclic relations, which generate the kernel of the mapping from $C_Y(T)$ onto B_n or D_n. There are four possible types of cyclic relations. Each type defines one of the classical affine Coxeter groups, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$, which are periodic permutations or signed permutation groups (see [2]). $\tilde{A}_n$ is the well-known $\tilde{S}_{n+1}$, where the period is $n+1$, which means $\tilde{A}_n$ is a periodic permutation group which satisfies $\pi(i+(n+1)) = \pi(i)+(n+1)$ for every permutation in $\tilde{A}_n$. The other three affine Coxeter groups are periodic

sign permutations with a period of $2n+2$, which satisfies $\pi(i+(2n+2)) = \pi(i)+(2n+2)$, and in addition $\pi(-i) = -\pi(i)$, where in the sequel, $-i$ will be denoted by $\bar{i}$, when we treat -1 in a signed permutation. It is well known that $\tilde{A}_n$ is isomorphic to $\mathbb{Z}^n \rtimes A_n$, or $\mathbb{Z}^n \rtimes S_{n+1}$. Similarly, $\tilde{B}_n$ is isomorphic to $\mathbb{Z}^n \rtimes B_n$, $\tilde{C}_n$ is isomorphic to $\mathbb{Z}^n \rtimes B_n$, $\tilde{D}_n$ is isomorphic to $\mathbb{Z}^n \rtimes B_n$, where $\mathbb{Z}^n$ is the group $A_{1,n}$ which will be defined in Section 6.

In Section 6 we define a group $A_{t,n}$ which will be used for the main theorem, and in Section 7 we prove the main theorem which states that $C_Y(T)$ is isomorphic to the semi-direct product of $A_{t,n}$ (which was defined in Section 6) by B_n or D_n, if the signed graph of $C(T)$ contains loops or does not contain loops, respectively.

2. The group $C(T)$

Let T' be a graph which contains $2n$ vertices $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. The edges which connect the vertices are defined as follows:

$$(x_i, x_j) \quad \text{is an edge} \iff (y_i, y_j) \quad \text{is an edge}$$

and

$$(x_i, y_j) \quad \text{is an edge} \iff (x_j, y_i) \quad \text{is an edge.}$$

For every $i \neq j$, a pair of edges $(x_i, x_j)(y_i, y_j)$ or $(x_i, y_j)(x_j, y_i)$ presents a generator of $C(T)$. For $i = j$, an edge (x_i, y_i) presents a generator of $C(T)$, see for example Figure 2.

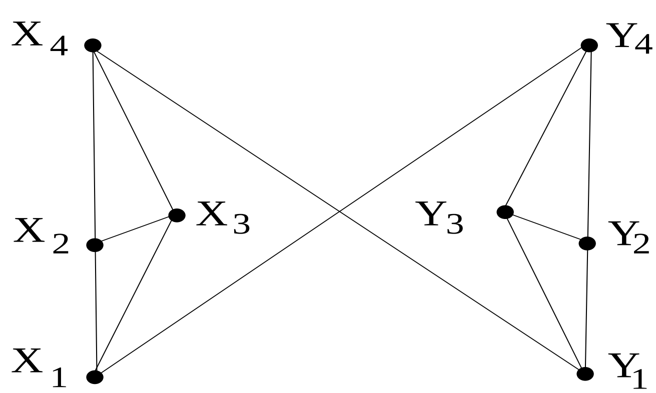
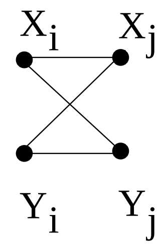


Figure 2. An example of a graph for $C(T)$

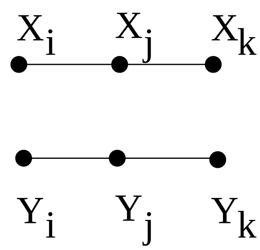
The group $C(T)$ admits the following relations on the edges:

(I). For distinct i, j, k (it is the case where two pairs of edges, which symbolize two generators, meet at two vertices):

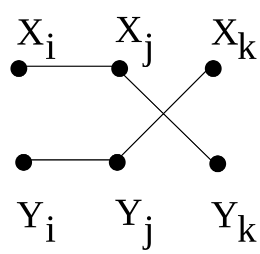
(1) $$((x_i, x_j)(y_i, y_j) \cdot (x_i, y_j)(x_j, y_i))^2 = 1,$$



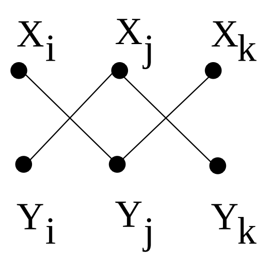
(2) $$((x_i, x_j)(y_i, y_j) \cdot (x_j, x_k)(y_j, y_k))^3 = 1,$$



(3) $$((x_i, x_j)(y_i, y_j) \cdot (x_j, y_k)(x_k, y_j))^3 = 1,$$

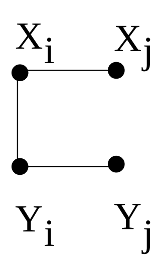


(4) $$((x_i, y_j)(x_j, y_i) \cdot (x_j, y_k)(x_k, y_j))^3 = 1,$$



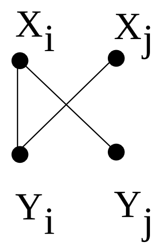
and a non simply-laced relation may hold if and only if there is a generator of the form (x_i, y_i), which admits (for distinct i and j):

(5) $$((x_i, y_i) \cdot (x_i, x_j)(y_i, y_j))^4 = 1,$$



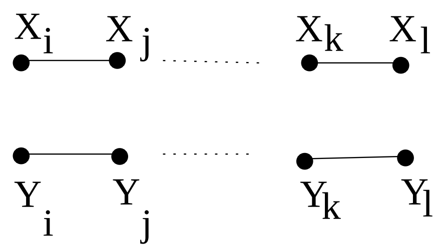
and

(6) $$((x_i, y_i) \cdot (x_i, y_j)(x_j, y_i))^4 = 1,$$

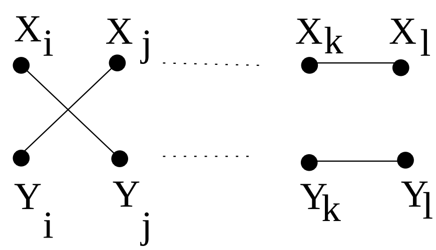


(II). For distinct i, j, k, l (it is the case where two pairs of edges are disjoint):

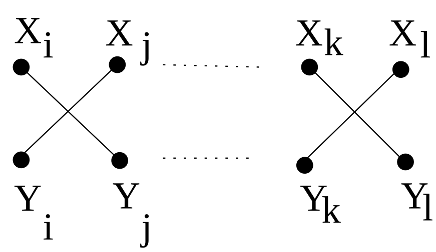
(7) $$((x_i, x_j)(y_i, y_j) \cdot (x_k, x_l)(y_k, y_l))^2 = 1,$$



(8) $$((x_i, y_j)(x_j, y_i) \cdot (x_k, x_l)(y_k, y_l))^2 = 1,$$

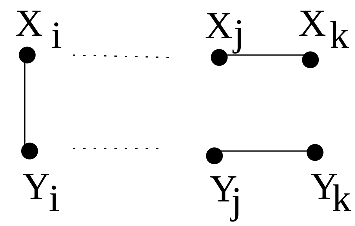


(9) $$((x_i, y_j)(x_j, y_i) \cdot (x_k, y_l)(x_l, y_k))^2 = 1,$$

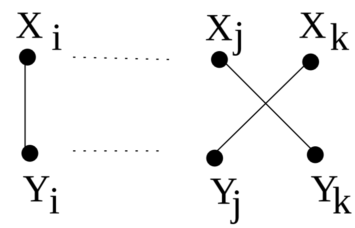


(III). For distinct i, j and k (it is the case where an edge (x_i, y_i) is disjoint from a pair of edges):

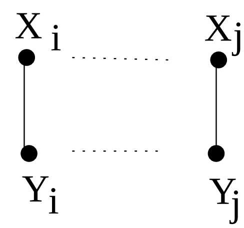
$$((x_i, y_i) \cdot (x_j, x_k)(y_j, y_k))^2 = 1, \tag{10}$$



$$((x_i, y_i) \cdot (x_j, y_k)(x_k, y_j))^2 = 1, \tag{11}$$



$$((x_i, y_i) \cdot (x_j, y_j))^2 = 1, \tag{12}$$



Each graph T' which satisfies the above described relations, has a natural mapping into B_n or D_n. Each pair of the form $(x_i, x_j)(y_i, y_j)$ is mapped to the element $(ij)(\bar{i}\bar{j})$; a pair of the form $(x_i, y_j)(x_j, y_i)$ is mapped to the element $(i\bar{j})(\bar{i}j)$; and an edge of the form (x_i, y_i) is mapped to the transposition $(i\bar{i})$. In the case that there are no edges of the form (x_i, y_i), the group $C(T)$ has a natural map into D_n (and $C(T)$ is simply-laced).

3. The reduced signed graphs

Due to the symmetry between x_i and y_i, we may consider an equivalent reduced signed graph T.

Instead of a graph T' with $2n$ vertices, we consider a signed graph T [1] with only n vertices, such that there are two types of edges, which connect the vertices. We replace $(x_i, x_j)(y_i, y_j)$ by $(x_i, x_j)_1$, and $(x_i, y_j)(x_j, y_i)$ by $(x_i, x_j)_{-1}$. We replace also (x_i, y_i) by a loop $(x_i, x_i)_{-1}$.

Then B_n and D_n are presented by graphs in Figure 3 (see original graphs in Figure 1 for comparison):

We note that the type of the group $C(T)$ in [5] can be presented as a graph, where all edges are of type 1. This is due to the existence of a natural mapping of $C(T)$ onto the symmetric group S_n.

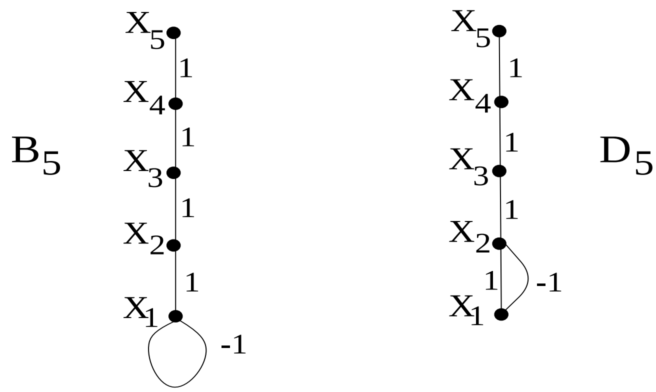


FIGURE 3.

Definition 1. *The edges $(x_i, x_j)_1$ and $(x_i, x_j)_{-1}$ are called conjugated edges.*

The relations which hold in the reduced graph are induced from the ones, which relate to the original graphs. For $a, b \in \{1, -1\}$ and for distinct i, j, k, l we have:
two conjugated edges commute

$$((x_i, x_j)_1 \cdot (x_i, x_j)_{-1})^2 = 1 \quad \text{derived from (1)}, \tag{13}$$

two edges meet at a vertex

$$((x_i, x_j)_a \cdot (x_j, x_k)_b)^3 = 1 \quad \text{derived from (2)-(4)}, \tag{14}$$

a loop and an edge meet at a vertex

$$((x_i, x_i)_{-1} \cdot (x_i, x_j)_a)^4 = 1 \quad \text{derived from (5)-(6)}, \tag{15}$$

two edges are disjoint

$$((x_i, x_j)_a \cdot (x_k, x_l)_b)^2 = 1 \quad \text{derived from (7)-(9)}, \tag{16}$$

a loop and an edge are disjoint

$$((x_i, x_i)_{-1} \cdot (x_j, x_k)_a)^2 = 1 \quad \text{derived from (10)-(11)}, \tag{17}$$

two loops are disjoint

$$((x_i, x_i)_{-1} \cdot (x_j, x_j)_{-1})^2 = 1 \quad \text{derived from (12)}. \tag{18}$$

In addition there is a relation which arises from cycles with odd number of edges, signed by -1, similarly to the relation which appears in [1, Page 193]. We call it an anti-cycle relation.

Definition 2. Anti-cycles

Let $x_1, \dots, x_n$ be n vertices on a cycle. The edges are

$$u_1 := (x_1, x_2)_{a_1}, \quad \dots, \quad u_{n-1} := (x_{n-1}, x_n)_{a_{n-1}}, \quad u_n := (x_n, x_1)_{a_n},$$

where $a_i \in \{1, -1\}$, $1 \leq i \leq n$ and $\#\{a_i \mid a_i = -1\}$ is odd.

In this case we have:

$$(u_1 u_2 \cdots u_{n-1} \cdot u_n u_{n-1} \cdots u_2)^2 = 1. \tag{19}$$

In a similar way, we derive relations of the form (for $1 \leq i \leq n$)

$$(u_i \cdot u_{i+1} \cdots u_n u_1 \cdots u_{i-1} u_{i-2} u_{i-3} \cdots u_1 u_n \cdots u_{i+1})^2 = 1. \tag{20}$$

Remark 3. *If a signed graph T does not contain any anti-cycle (even no conjugated edges, which is an anti-cycle of length two) neither a loop, then the graph T describes the same groups which appears in* [5]*, where the natural homomorphism is by omiting the signs. It is homomorphism, since the additional relation which described in this paper caused by anti-cycle relations (including conjugated edges) or by relations involving loops. Hence, we assume that T contains at least one anti-cycle or a loop (otherwise the result is in* [5]*).*

There are graphs T where this additional relation makes $C(T)$ to be a generalized Coxeter Group as it appears in [1]. For example, in Figure 4 one can find a group, which is a generalized one, since we have an anti-cycle and a cycle, which contain the same three vertices.

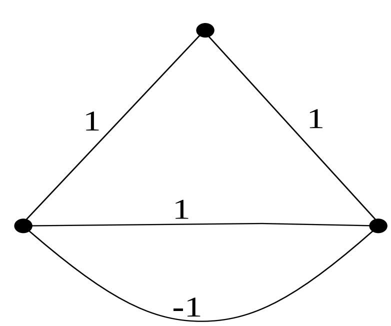


FIGURE 4.

Remark 4. *We notice that the most simple case for an anti-cycle are two conjugated edges* $u_1 = (x_1, x_2)_1$ *and* $u_2 = (x_1, x_2)_{-1}$ *which form an anti-cycle. Then the relation is just saying* u_1 *commutes with* u_2 *which we have already assumed (see Relation (13)).*

Lemma 5. *Let* T *be a connected signed graph with* n *vertices* $x_1, \cdots x_n$*, and let* $\phi : C(T) \to B_n$ *the natural mapping such that* $\phi((x_i x_j)_1) = (ij)(\bar{i}\bar{j})$, $\phi((x_i x_j)_{-1}) = (\bar{i}j)(i\bar{j})$*, and* $\phi((x_i x_i)_{-1}) = (i\bar{i})$ *for every* $1 \leq i, j \leq n$*. Then the following holds:*

1) If T *does not contain a loop nor an anti-cycle then* $Im(\phi)$ *is a subgroup of* B_n *isomorphic to* S_n*.*

2) If T *does contain an anti-cycle but does not contain a loop, then* $Im(\phi) = D_n$*.*

3) If T *does contain a loop then* $Im(\phi) = B_n$*.*

We use three propositions to prove the lemma.

Proposition 6. *Let* $x_1 \cdots x_k$ *be* k *vertices in an anti-cycle, where the edges are* $w_i := (x_{i-1}x_i)_{a_{i-1}}$ *and* $w_1 := (x_k x_1)_{a_k}$*. Then*

$\phi(w_{i+1}w_{i+2}\cdots w_k w_1 \cdots w_{i-2}w_{i-1}w_{i-2}\cdots w_1 w_k \cdots w_{i+1}) = (i-1, i)(\overline{i-1}, \bar{i})$*, in case* $a_{i-1} = -1$ *which means,* $\phi(w_i) = (\overline{i-1}, i)(i-1, \bar{i})$*.*

$\phi(w_{i+1}w_{i+2}\cdots w_k w_1 \cdots w_{i-2}w_{i-1}w_{i-2}\cdots w_1 w_k \cdots w_{i+1}) = (\overline{i-1}, i)(i-1, \bar{i})$ *in case* $a_{i-1} = 1$ *which means,* $\phi(w_i) = (i-1, i)(\overline{i-1}, \bar{i})$*.*

Proposition 7. *Let be a signed path connected to an anti-cycle, where* $x_1 \cdots x_k$ *be* k *vertices in an anti-cycle, and the edges are* $w_i := (x_{i-1}x_i)_{a_{i-1}}$ *and* $w_1 := (x_k x_1)_{a_k}$ *and the vertices of the path are* $x_k, \cdots, x_s$ *and the connecting edges are* $w_i := (x_{i-1}x_i)_{a_i}$ *for* $k+1 \leq i \leq s$*. Then*

$\phi(w_i^{w_{i-1}\cdots w_{k+1}w_k\cdots w_2\cdots w_k w_1 w_{k+1}\cdots w_{i-1}}) = (i-1, i)(\overline{i-1}, \bar{i})$ *in case* $a_{i-1} = -1$ *which means,* $\phi(w_i) = (\overline{i-1}, i)(i-1, \bar{i})$*. and*

$\phi(w_i^{w_{i-1}\cdots w_{k+1}w_k\cdots w_2\cdots w_k w_1 w_{k+1}\cdots w_{i-1}}) = (\overline{i-1}, i)(i-1, \bar{i})$ *in case* $a_{i-1} = 1$ *which means,* $\phi(w_i) = (i-1, i)(\overline{i-1}, \bar{i})$*.*

where a^b *means* a *conjugated by* b*.*

Proposition 8. *Let be a signed path connected to a loop, where* x_0 *is a vertex containing a loop* v*, and* $w_i := (x_{i-1}x_i)_{a_{i-1}}$ *are the vertices of a path. Then*

$\phi(w_{i-1}\cdots w_1 v w_1 \cdots w_{i-1} w_i w_{i-1} \cdots w_1 v w_1 \cdots w_{i-1}) = (i-1, i)(\overline{i-1}, \bar{i})$ *in case* $a_{i-1} = -1$ *which means,* $\phi(w_i) = (\overline{i-1}, i)(i-1, \bar{i})$*.*

$\phi(w_{i-1}\cdots w_1 v w_1 \cdots w_{i-1} w_i w_{i-1} \cdots w_1 v w_1 \cdots w_{i-1}) = (i-1,i)(\overline{i-1},\bar{i})$ *in case* $a_{i-1} = 1$ *which means,* $\phi(w_i) = (i-1,i)(\overline{i-1},\bar{i})$.

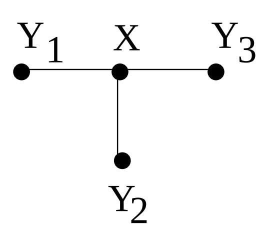
Proof of Lemma 5 Assume 1) holds. Then T does not contain a loop nor an anti-cycle, then by omiting the signs of T, mapping the edges onto S_n (remark 3).

Assume 2) holds. Since T is connected and contains at least one anti-cycle, every edge in T either lies on an anti-cycle or connected by a path to an anti-cycle. Hence, if $\phi((x_i x_j)_1) = (ij)(\bar{i}\bar{j})$, then by Propostions 6 and 7 there exists an element $w \in C(T)$ such that $\phi(w) = (\bar{i}j)(i\bar{j})$. On the other hand, if $\phi((x_i x_j)_{-1}) = (\bar{i}j)(i\bar{j})$ then by the same argument there exists w such that $\phi(w) = (ij)(\bar{i}\bar{j})$. Since T is connected, there is a path connecting any two vertices in T, then by the same argument as in [5] for every distinct i and j such that $1 \leq i, j \leq n$, there are elements w_1 and w_2 such that $\phi(w_1) = (ij)(\bar{i}\bar{j})$ and $\phi(w_2) = (\bar{i}j)(i\bar{j})$. The subgroup of B_n which is generated by all signed transpositions is D_n.

Assume 3 holds. Since T is connected and contains a loop, every edge which is not a loop connected with a path to a loop. Hence, if $\phi((x_i x_j)_1) = (ij)(\bar{i}\bar{j})$, then by Propostion 8 there exists an element $w \in C(T)$ such that $\phi(w) = (\bar{i}j)(i\bar{j})$. On the other hand, if $\phi((x_i x_j)_{-1}) = (\bar{i}j)(i\bar{j})$ then by the same argument there exists w such that $\phi(w) = (ij)(\bar{i}\bar{j})$. Since T contains a loop, then there exists an element v such that $\phi(v) = (i\bar{i})$, and the subgroup of B_n which is generated by allthe signed transpositions $(ij)(\bar{i}\bar{j})$, $(\bar{i}j)(i\bar{j})$ and an element of a form $(i\bar{i})$ is all B_n.

4. The group $C_Y(T)$

We define the group $C_Y(T)$ as a quotient of $C(T)$ by the 'fork' relations. The fork relations in $C(T)$ are (for $a, b, c \in \{1, -1\}$):

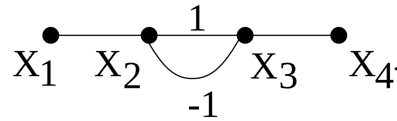
I. Three edges meet at a common vertex: 

.

$$((x, y_1)_a \cdot (x, y_2)_b)^3 = ((x, y_1)_a \cdot (x, y_3)_c)^3 = ((x, y_2)_b \cdot (x, y_3)_c)^3 = 1.$$

Then (R_1) is (as in [5]):

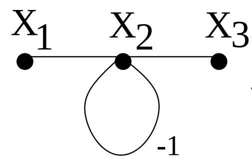
$$((x, y_1)_a \cdot (x, y_2)_b (x, y_3)_c (x, y_2)_b)^2 = 1. \tag{21}$$

II. Two conjugated edges $(x_2, x_3)_1$ and $(x_2, x_3)_{-1}$ meet at both of their common vertices (x_2 and x_3), two other edges $(x_1, x_2)_a$ and $(x_3, x_4)_b$ 

.

Then (R_2) is:

$$((x_1, x_2)_a(x_2, x_3)_1(x_1, x_2)_a \cdot (x_3, x_4)_b(x_2, x_3)_{-1}(x_3, x_4)_b)^2 = 1. \tag{22}$$

III. A loop and two edges meet at a vertex 

.

Then (R_3) is:

$$(R_3)((x_2, x_2)_{-1} \cdot (x_1, x_2)_a(x_2, x_3)_b(x_1, x_2)_a)^2 = 1, \tag{23}$$

and (R_4) is:

$$((x_1, x_2)_a \cdot (x_2, x_2)_{-1}(x_2, x_3)_b(x_2, x_2)_{-1})^3 = 1. \tag{24}$$

We recall that in order to prove these relations, we consider u_i as a signed permutation in B_n, where $(x_i, x_{i+1})_1$ is $(i, i+1)(\bar{i}, \overline{i+1})$ and $(x_i, x_{i+1})_{-1}$ is $(i, \overline{i+1})(\bar{i}, i+1)$.

Note that in the case of D-covers, we may have only (21) and (22), since (23) and (24) involve loops, which may appear only in B-covers. Thus:

$$C_Y(T) = C(T)/\langle (21) \cup (22) \rangle \quad \text{for D-covers}$$

and

$$C_Y(T) = C(T)/\langle (21) \cup (22) \cup (23) \cup (24) \rangle \quad \text{for B-covers.}$$

5. Mapping $C_Y(T)$ onto B_n or D_n

Now we classify the relations, which may appear in the kernel of the mapping from $C_Y(T)$ onto B_n or D_n (similarly as done for the 'cyclic' relations in [5]).

(I). Cycles:

Let T be connected signed graph which contains at least one anti-cycle. Let $x_0, \dots, x_{m-1}$ be m vertices on a cycle, which are connected by the m edges $(x_{i-1}, x_i)_{a_{i-1}}$, and $(x_{m-1}, x_0)_{a_{m-1}}$ where $\#\{a_i \mid a_i = -1\}$ is even.

If $a_{i-1} = 1$, then $u_i := (x_{i-1}x_i)_{a_{i-1}}$.

If $a_{i-1} = -1$, then $\bar{u}_i := (x_{i-1}x_i)_{a_{i-1}}$.

Now define u_i for the cases where $a_{i-1} = -1$, and $\bar{u}_i$ for the cases where $a_{i-1} = 1$. Since, T is connected and T does contain an anti-cycle or a loop, let $w_1, \cdots w_k$ be k edges which form an anti-cycle of length k in case T contains an anti-cycle, otherwise, lew w be a loop. Let $v_1, \cdots v_s$ be a path connecting the anti-cycle of length k or the loop with the cycle of length m. Then:

In case $a_0 = -1$:

$$u_1 := \bar{u}_1^{v_s \cdots v_1 w_1 w_k^{w_{k-1} \cdots w_2} v_1 \cdots v_s} \tag{24}$$

and in case $a_0 = 1$:

$$\bar{u}_1 := u_1^{v_s \cdots v_1 w_1 w_k^{w_{k-1} \cdots w_2} v_1 \cdots v_s} \tag{24}$$

Then inductively we define u_i where $a_{i-1} = -1$ and $\bar{u}_i$ where $a_{i-1} = 1$ for every $1 \leq i \leq m$ as following

$$u_i := \bar{u}_i^{u_{i-1} \cdots u_1 v_s \cdots v_1 w_1 w_k^{w_{k-1} \cdots w_2} v_1 \cdots v_s u_1 \cdots u_{i-1}}. \tag{24}$$

$$\bar{u}_i := u_i^{u_{i-1} \cdots u_1 v_s \cdots v_1 w_1 w_k^{w_{k-1} \cdots w_2} v_1 \cdots v_s u_1 \cdots u_{i-1}} \tag{24}$$

where we denote a^b instead of $b^{-1}ab$.

In case of loop instead of anti-cycle, we write w instead of $w_1 w_k^{w_{k-1} \cdots w_2}$ in equations 5, 5, 5 and 5.

Remark 9. *We notice that the existence of an anti-cycle or a loop connecting the cycle allows us to define u_i and $\bar{u}_i$ for every $1 \leq i \leq n$, such that the natural mapping ϕ from $C(T)$ onto B_n or D_n satisfies*

$\phi(u_i) = (i-1, i)(\overline{i-1}, \bar{i})$ *and* $\phi(\bar{u}_i) = (i-1, \bar{i})(\overline{i-1}, i)$, $\phi(u_m) = (m-1, 0)(\overline{m-1}, \bar{0})$ *and* $\phi(\bar{u}_m) = (m-1, \bar{0})(\overline{m-1}, 0)$.

(II). $\tilde{D}$-type cycles:

Two anti-cycles connected by a path are called a $\tilde{D}$-type cycle. The length of the anti-cycles can be every length ≥ 2 (anti-cycle of length 2 means two conjugated edges). Let $x_0, \ldots, x_{m-1}$ be m vertices, where $x_0, \ldots, x_{k_1-1}$ form an anti-cycle of length k_1, and $x_{k_2}, \ldots, x_{m-1}$ form another anti-cycle, and there is a simple signed path connecting the vertices x_{k_1-1} and x_{k_2}. We define u_i and $\bar{u}_i$ for every $1 \leq i \leq m-1$. In case $k_1 = 2$: $u_1 := (x_0, x_1)_1$, $\bar{u}_1 := (x_0, x_1)_{-1}$, otherwise we look at the sign of the edge connecting x_0 and x_1. If the sign is $+1$ then $u_1 := (x_0, x_1)_1$ and $\bar{u}_1 := (x_{k_1-1}, x_0)_{a_{k_1-1}}$ conjugated by $(x_1, x_2)_{a_1}(x_2, x_3)_{a_2} \cdots (x_{k_1-2}, x_{k_1-1})_{a_{k_1-2}}$, where a_i is the sign of the edge connecting x_i with x_{i+1}. If the sign of the edge connecting x_0 with x_1 is -1, then $\bar{u}_1 := (x_0, x_1)_{-1}$ and $u_1 := (x_{k_1-1}, x_0)_{a_{k_1-1}}$ conjugated by $(x_1, x_2)_{a_1}(x_2, x_3)_{a_2} \cdots (x_{k_1-2}, x_{k_1-1})_{a_{k_1-2}}$.

Similarly, we define u_{m-1} and $\bar{u}_{m-1}$ where we look at the second anti-cycle. If the length of the second anti-cycle is 2, then:

$$u_{m-1} := (x_{m-2}, x_{m-1})_1, \quad \bar{u}_{m-1} := (x_{m-2}, x_{m-1})_{-1},$$

otherwise, similarly to the definition of u_1 and $\bar{u}_1$ we look at the sign of the edge connecting x_{m-2} and x_{m-1}. If the sign is $+1$ then $u_{m-1} := (x_{m-2}, x_{m-1})_1$ and $\bar{u}_{m-1} := (x_{k_2}, x_{m-1})_{a_{m-1}}$ conjugated by $(x_{m-1}, x_{m-2})_{a_{m-2}}(x_{m-2}, x_{m-3})_{a_{m-3}} \cdots (x_{k_2+1}, x_{k_2})_{a_{k_2}}$, where a_i is the sign of the edge connecting x_i with x_{i+1}. If the sign of the edge connecting x_0 with x_1 is -1, then $\bar{u}_{m-1} := (x_{m-2}, x_{m-1})_{-1}$ and $u_{m-1} := (x_{k_2}, x_{m-1})_{a_{m-1}}$ conjugated by $(x_{m-1}, x_{m-2})_{a_{m-2}}(x_{m-2}, x_{m-3})_{a_{m-3}} \cdots (x_{k_2+1}, x_{k_2})_{a_{k_2}}$.

And we define u_i and $\bar{u}_i$ for every $2 \leq i \leq m-2$ in the following way. We denote an edge in the signed graph (for $2 \leq i \leq m-2$) as $(x_{i-1}, x_i)_{a_i}$.

If $a_i = 1$, then:

$$u_i := (x_{i-1}, x_i)_1$$

and

$$\bar{u}_i := u_{i-1}u_{i-2} \cdots u_2 u_1 \bar{u}_1 u_2 \cdots u_{i-1} u_i u_{i-1} \cdots u_2 u_1 \bar{u}_1 u_2 \cdots u_{i-2} u_{i-1}.$$

If $a_i = -1$, then:

$$\bar{u}_i := (x_{i-1}, x_i)_{-1},$$

and

$$u_i := u_{i-1}u_{i-2}\cdots u_2u_1\bar{u}_1u_2\cdots u_{i-1}\bar{u}_iu_{i-1}\cdots u_2u_1\bar{u}_1u_2\cdots u_{i-2}u_{i-1}.$$

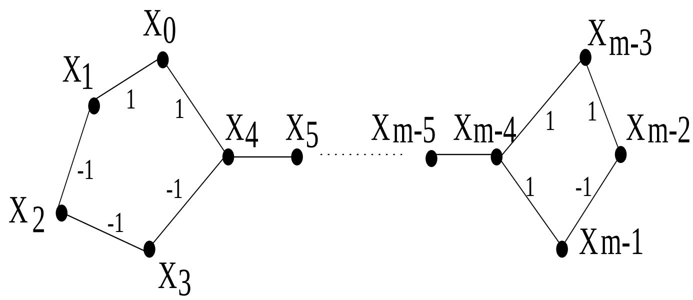


FIGURE 5. $\tilde{D}$-type cycle

Moreover, we define elements u_m and $\bar{u}_m$ to be

$$u_m := \bar{u}_1u_2u_3\cdots u_{m-2}\bar{u}_{m-1}u_{m-2}\cdots u_3u_2\bar{u}_1$$

and

$$\bar{u}_m := u_1u_2u_3\cdots u_{m-2}\bar{u}_{m-1}u_{m-2}\cdots u_3u_2u_1.$$

(III). $\tilde{B}$-type cycles:

A loop and an anti-cycle which are connected by a path are called $\tilde{B}$-type cycle. The length of the anti-cycles can be every length ≥ 2.

Let $x_0, \ldots, x_{m-1}$ be m vertices, where we have a loop in x_0, an anti-cycle connecting the vertices x_k and x_{m-1}, and a simple signed path between x_0 and x_k.

We define u_i and $\bar{u}_i$ in the following way (for $1 \leq i \leq m-1$):

Let $v := (x_0, x_0)_{-1}$. If $(x_0, x_1)_1$ belongs to the signed graph, then $u_1 := (x_0, x_1)_1$ and $\bar{u}_1 := vu_1v$. Otherwise, for $(x_0, x_1)_{-1}$ belonging to the signed graph, $\bar{u}_1 := (x_0, x_1)_{-1}$ and $u_1 := v\bar{u}_1v$.

For $2 \leq i \leq m-1$, we define u_i in the same way as it was defined for $\tilde{D}$-type cycles.

Moreover, we define elements u_m and $\bar{u}_m$ as follows:

$$u_m := vu_1u_2\cdots u_{m-2}\bar{u}_{m-1}u_{m-2}\cdots u_2u_1v$$

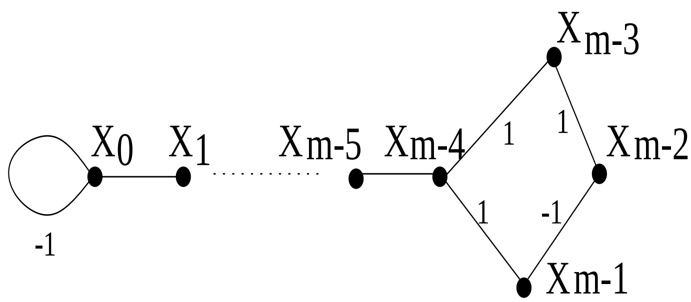


FIGURE 6. $\tilde{B}$-type cycle

and

$$\bar{u}_m := u_1 u_2 \cdots u_{m-2} \bar{u}_{m-1} u_{m-2} \cdots u_2 u_1.$$

Proposition 10. *Consider the natural mapping ϕ from the $\tilde{D}$-type or $\tilde{B}$-type cycle of length m onto D_m, or B_m $\phi(u_i) = (i-1, i)(\overline{i-1}, \bar{i})$ and $\phi(\bar{u}_i) = (i-1, \bar{i})(\overline{i-1}, i)$, $\phi(u_m) = (m-1, 0)(\overline{m-1}, \bar{0})$ and $\phi(\bar{u}_m) = (m-1, \bar{0})(\overline{m-1}, 0)$*

Remark 11. *We notice that there is an edge on $\tilde{D}$-type or on $\tilde{B}$-type cycle which one u_{m-1} admits only from the defined u_i and $\bar{u}_i$ (The edge connecting x_{m-2} to x_{m-1} or the edge connecting x_{m-1} to x_{m-4} depends on a_{m-2}). This edge will be important when we define the spanning 'tree' in section 7, where we omit this edge. Hence, by omiting this edge we omit u_{m-1} only.*

By symmetry we can define u_i in diferent way, such there will be one edge only in one of the anti-cycles such that one of the u_i's admits only, and $\phi(u_i)$ satisfies the condinitions of Proposition 10.

(IV). $\tilde{C}$-type cycles:

Two loops connected by a simple path are called a $\tilde{C}$-type cycle. Let $x_0, \ldots, x_{m-1}$ be m vertices, and two loops

$$v := (x_0, x_0)_{-1}, \qquad w := (x_{m-1}, x_{m-1})_{-1}.$$

We define u_i in the same way as it was defined for $\tilde{B}$-type cycles $(1 \leq i \leq m-1)$.

In addition we define elements u_m and $\bar{u}_m$ in the following way:

$$u_m := u_1 u_2 \cdots u_{m-2} u_{m-1} u_{m-2} \cdots u_2 u_1$$

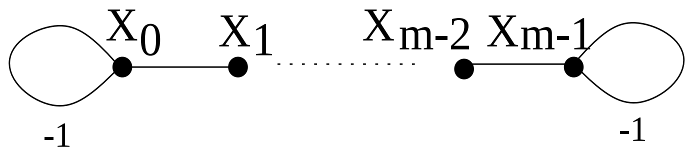


FIGURE 7. $\tilde{C}$-type cycle

and

$$\bar{u}_m := u_1u_2\cdots u_{m-2}wu_{m-1}wu_{m-2}\cdots u_2u_1.$$

Remark 12. *Propositon 10 holds for the natural mapping from $\tilde{C}$-type cycle of length m onto B_m too. We notice that from the defined elements u_i and $\bar{u}_i$, the element $\bar{u_m}$ admits only the loop w, where in the spanning 'tree' (will be defined in section 7 we omit this loop, hence omiting again $\bar{u}_m$ only from the defined u_i's and $\bar{u}_i$'s*

Proposition 13. *Let ϕ be the natural map from one of the cycle onto B_n or D_n. Then:*

$$\phi(u_1u_2u_3\cdots u_{m-1}) = \phi(u_2u_3u_4\cdots u_m)$$

and

$$\phi(u_mu_{m-1}\bar{u}_{m-1}u_mu_1u_2u_3\cdots u_{m-3}\bar{u}_{m-2}u_{m-1}) = \phi(u_1u_m\bar{u}_mu_1u_2u_3u_4\cdots u_{m-2}\bar{u}_{m-1}u_m).$$

Proof. The first equation has been proved in [5], and the second one we get easily by substituting the signed permutation $\phi(u_i)$ where ϕ is the natural map from $C_Y(T)$ onto B_n or D_n. By Proposition 10 for $1 \leq i \leq m-1$, $\phi(u_i) = (i-1, i)(\overline{i-1}, \bar{i})$ and $\phi(\bar{u_i}) = (\overline{i-1}, i)(i-1, \bar{i})$, and $\phi(u_m) = (m-1, 0)(\overline{m-1}, \bar{0})$, $\phi(\bar{u_m} = (\overline{m-1}, 0)(m-1, \bar{0})$. Then:

$$\begin{aligned}\phi(u_mu_{m-1}\bar{u}_{m-1}u_mu_1u_2u_3\cdots u_{m-3}\bar{u}_{m-2}u_{m-1}) &= \phi(u_1u_m\bar{u}_mu_1u_2u_3u_4\cdots u_{m-2}\bar{u}_{m-1}u_m) \\ &= (m-1\ \ m-2\ldots 0)(\overline{m-1}\ \ \overline{m-2}\ldots\bar{0}).\end{aligned}$$

□

The definition of u_i and $\bar{u}_i$ for $1 \leq i \leq m$ are important, since it enables defining γ_i and $\gamma_{\bar{i}}$ for every $1 \leq i \leq m$ for every type ($\tilde{A}$ or $\tilde{B}$ or $\tilde{C}$ or $\tilde{D}$) of cycle which contains n vertices, as defined in [5] (see Section 5).

We call the above figures $\tilde{B}$-, $\tilde{C}$- and $\tilde{D}$-types cycles, since the groups which are described by them are the affine groups $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$. By [4], an infinite Coxeter group is large if and only if the group is not affine. Hence, a diagram T defines a large group $C_Y(T)$ (quotient of

$C(T)$ by one of the relations which are mentioned in Section 4) for every graph other than one of the cycles which are mentioned here, an anti-cycle (which is a graph of D_n), a line connecting an anti-cycle or a loop. In Section 7 we will conclude that $C_Y(T)$ is large if and only if T does contain at least two cycles.

6. The group $A_{t,n}$

Similarly to [5], we define a group $A_{t,n}$. Let $X = \{x, y, z, \dots\}$ be a set of size t and $R = \{r_x, r_y, \dots\}$ be a set of size t_1, where $t_1 \leq t$, and the indices of the r's are in a subset of X.

Definition 14. *The group $A_{t,n}$ is generated by $(2n)^2|X| + 2n|R|$ elements x_{ij}, and r_{x_k} where $x \in X, \quad r \in R \quad i, j, k \in \{1, 2, \dots, n, \bar{1}, \bar{2}, \dots, \bar{n}\}$ and $\bar{\bar{i}} = i$ (we write $\bar{i}$ instead of $-i$).*

$$x_{ii} = 1 \tag{25}$$

$$x_{ij}^{-1} = x_{ji} \tag{26}$$

$$x_{ij}x_{jk} = x_{jk}x_{ij} = x_{ik} \quad \textit{for every } i,\ j \textit{ and } k \tag{27}$$

$$r_{x,i}r_{x,j} = x_{i\bar{j}} \quad \textit{for every } i \textit{ and } j \tag{28}$$

$$x_{ij}y_{kl} = y_{kl}x_{ij} \quad \textit{and} \quad x_{ij}x_{kl} = x_{kl}x_{ij} \quad \textit{for every distinct } i, j, k, l \tag{29}$$

and in addition

$$x_{\bar{j}\bar{i}} = x_{ij} \tag{30}$$

$$x_{\bar{i}j}y_{\bar{j}k}x_{\bar{k}\bar{i}}y_{i\bar{j}}x_{j\bar{k}}y_{k\bar{i}} = 1 \tag{31}$$

$$r_{x,i}y_{\bar{i}j}r_{x,j}r_{x,k}y_{\bar{k}\bar{i}}r_{x,\bar{i}}r_{x,\bar{j}}y_{j\bar{k}}r_{x,\bar{k}} = 1 \tag{32}$$

$$r_{x,i}y_{\bar{i}j}r_{x,j}z_{\bar{j}k}r_{x,k}y_{\bar{k}\bar{i}}r_{x,\bar{i}}z_{i\bar{j}}r_{x,\bar{j}}y_{j\bar{k}}r_{x,\bar{k}}z_{ki} = 1. \tag{33}$$

Proposition 15. *For $n \geq 5$ or $t \leq 2$ the following (from [5]) hold in $A_{t,n}$:*

$$[w_{is}, x_{jk}y_{kl}x_{kj}] = 1 \quad \textit{for distinct } i, j, k, l, s \tag{34}$$

$$x_{si}y_{ij}x_{js}w_{sk} = w_{sk}x_{ki}y_{ij}x_{jk} \quad \textit{for distinct } i, j, k, s \tag{35}$$

$$x_{si}y_{ij}x_{js} = x_{ki}y_{ij}x_{jk} \quad \textit{for distinct } i, j, k, s \tag{36}$$

$$[x_{si}y_{ij}x_{js}, u_{jl}v_{l\bar{s}}u_{\bar{s}j}] = 1 \quad \textit{for } t \leq 2 \textit{ or } n \geq 6. \tag{37}$$

Proof. Relations (34) and (35) are proved in [5].

We prove Relation (36). Let us consider the relation $x_{si}y_{ij}x_{js}w_{sk}x_{kj}y_{ji}x_{ik}w_{ks} = 1$. By Relation (34), this relation becomes $x_{si}y_{ij}x_{js}x_{kj}y_{ji}x_{jk}w_{sk}x_{ij}w_{ks} = 1$, and we are able to omit w_{ks} and w_{sk} (since by Relation (29), $w_{sk}x_{ij}w_{ks} = x_{ij}w_{sk}w_{ks} = x_{ij}$). Therefore we get $x_{si}y_{ij}x_{js}x_{kj}y_{ji}x_{jk}x_{ij} = 1$, and this gives us $(x_{si}y_{ij}x_{js})(x_{kj}y_{ji}x_{ik}) = 1$ (which is exactly (36)).

Now we prove Relation (37). If $n \geq 6$, there exist t and k, distinct from i, j, s, l, such that $x_{si}y_{ij}x_{js} = x_{ti}y_{ij}x_{jt}$ and $u_{jl}v_{l\bar{s}}u_{\bar{s}j} = u_{kl}v_{l\bar{s}}u_{\bar{s}k}$ (by 36). And we can conclude that $[x_{ti}y_{ij}x_{jt}, u_{kl}v_{l\bar{s}}u_{\bar{s}k}] = 1$ for $t \leq 2$ or $n \geq 6$. □

It is possible to define an action of B_n on $A_{t,n}$ as follows: $\sigma^{-1}x_{ij}\sigma := x_{\sigma(i)\sigma(j)}$ and $\sigma^{-1}r_{x,i}\sigma := r_{x,\sigma(i)}$ for every $\sigma \in B_n$ (similarly to the action of S_n in [5]).

The $A_{t,n}$ has t Abelian subgroups $Ab(x)$, where $Ab(x)$ is: x_{ij}, $x_{\bar{i}j}$ for a particular x or x_{ij}, $x_{\bar{i}j}$ and $r_{x,k}$ for a particular x (where $r_{x,k}$ exists for the specific x and $1 \leq i, j, k \leq n$). We see that the described groups $Ab(x)$ are abelian by using Relations (27), (28), (29) and (30).

Each subgroup $Ab(x)$ is freely generated by n elements $x_{i,i+1}$ (where $1 \leq i \leq n-1$) and $x_{\bar{1}1}$ if $r_{x,j}$ does not exists. If $r_{x,j}$ exists, $Ab(x)$ is freely generated by the n elements $x_{i,i+1}$ (where $1 \leq i \leq n-1$) and $r_{x,1}$. In [5, page 13] it has been shown that the subgroup x_{ij}, where $1 \leq i, j \leq n$ is freely generated by the set $x_{i,i+1}$, where $1 \leq i \leq n-1$. Using Relation (27), $x_{\bar{i}j} = x_{\bar{i}\bar{1}}x_{\bar{1}1}x_{1j}$. Then using Relation (30), $_{\bar{i}j} = x_{1i}x_{\bar{1}1}x_{1j}$. Hence, adding a generator $x_{\bar{1}1}$, we get all the elements $x_{ij} \cup x_{\bar{i}j}$, where $1 \leq i, j \leq n$. In case where $r_{x,i}$ exists, by using Relation (28), $r_{x_1}^2 = x_{1\bar{1}}$. Then using Relation (26), $x_{1\bar{1}} = x_{\bar{1}1}^{-1}$. Hence, for x where $r_{x,i}$ exists, $Ab(x)$ is freely generated by $x_{i,i+1}$ and $r_{x,1}$, where $1 \leq i \leq n-1$.

7. The Main Theorem

Theorem 16. *Assume there is at least one anti-cycle or a loop in T. Then the group $C_Y(T)$ is isomorphic to $A_{t,n} \rtimes D_n$ if there are no loops in T. In the case of the existence of loops in T, it is isomorphic to $A_{t,n} \rtimes B_n$.*

In order to prove the theorem, we define, as in [5], a spanning 'tree' T_0. Note that for us, 'tree' means that T_0 is connected and there are no cycles of any type in T_0 (no cycles of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$-type). But we allow the existence of anti-cycles (cycles with odd number of

edges, signed -1), and in particular we allow loops and two conjugate edges to connect two vertices (which is an anti-cycle of length 2).

Now we explain how we get the spanning 'tree' from the signed graph of $C(T)$: In case of $\tilde{A}$-type, we get T_0 by omitting one arbitrary edge, as it occurs in [5]. In case of $\tilde{D}$-type or $\tilde{B}$-type cycle, omitting one of the edges in one of the anti-cycles (see Figures 5 and 6). In case of cycles of $\tilde{C}$-type, omitting one of the loops v or w (see Figure 7).

We define γ_i and $\gamma_{\bar{i}}$ for $1 \leq i \leq n$. In case of $\tilde{C}$-type cycle, where we omit a loop to get the spanning tree, we define δ_i and $\delta_{\bar{i}}$ too.

We have already defined edges u_i and $\bar{u}_i$, for every $1 \leq i \leq m$ in $\tilde{B}$, $\tilde{C}$, $\tilde{D}$-type cycles with m vertices. So, we can define certain elements γ_i, $\gamma_{\bar{i}}$, δ_i and $\delta_{\bar{i}}$ in every cycle in T:

$$\gamma_i : \ = \ u_{i+2}u_{i+3}\cdots u_m u_1 \cdots u_i \tag{38}$$

$$\gamma_{\bar{i}} : \ = \ u_{i+1}u_i\bar{u}_i u_{i+1}u_{i+2}\cdots u_m u_1 \cdots \bar{u}_{i-1}u_i \tag{39}$$

$$\delta_i : \ = \ u_i u_{i-1}\cdots u_1 v u_1 u_2 \cdots u_{m-1} w u_{m-1} u_{m-2} \cdots u_{i+1} \tag{40}$$

$$\delta_{\bar{i}} : \ = \ \delta_i^{-1} \tag{41}$$

for every $1 \leq i \leq m$ and every $\tilde{C}$-type cycle of length m in T.

Note that the definition of γ_i for $i > 0$ is the same as in [5]. In addition, we define $\gamma_{\bar{i}}$ too, which has not been defined before. The following property is important for the main theorem:

Proposition 17. $\gamma_{\bar{i}}^{-1}\gamma_{\bar{j}} = \gamma_j^{-1}\gamma_i$ *for every i and j.*

Proposition 18. $[\gamma_i^{-1}\gamma_j, \gamma_k^{-1}\gamma_l] = 1$ *for every $i, j, k, l \in \{1, 2, \cdots m, \bar{1}, \bar{2}, \cdots, \bar{m}\}$ (i, j, k and l are not necessarily distinct).*

Proposition 19. $[\delta_i, \gamma_i^{-1}\gamma_j] = 1$ *and* $\delta_i\delta_j = \gamma_{\bar{j}}^{-1}\gamma_i$.

Proof of Propositions 17, 18 and 19:

The proof is by looking at the elements γ_i (as it defined) in the affine groups $\tilde{B}_m$, $\tilde{C}_m$ or $\tilde{D}_m$ as periodic signed permutations with a period of $2m+2$, which means $\pi(i + (2m + 2)) = \pi(i) + (2m + 2)$ for every i [2]. Then for $j \neq \bar{i}$, the element $\gamma_j^{-1}\gamma_i$ is the periodic signed permutation π which satisfies $\pi(i) = i + (2m + 2)$, $\pi(j) = j - (2m + 2)$, $\gamma_{\bar{i}}^{-1}\gamma_i$ is the periodic

signed permutation $\pi(i) = i + 2 * (2m+2)$ and δ_i is the periodic signed permutation in $\tilde{C}_m$ which satisfies $\pi(i) = i + (2m+2)$. Since, $\pi(i) = i + (2m+2)$ means $\pi(-i) = -i - (2m+2)$ and $pi(j) = j - (2m+2)$ means $\pi(-j) = -j + (2m-2)$, Proposition 17 holds. Propositions 18 and 19 hold, since every two periodic permutations π and τ in an affine group $\tilde{B}_m$, $\tilde{C}_m$ or $\tilde{D}_m$ commutes where $\pi(i) = i + k(2nm+2)$, for every i and some $k \in \mathbb{Z}$.

Proposition 20. *T is a connected signed graph T with n vertices, then it is possible to extend the definition of γ_i and $\gamma_{\bar{i}}$ for every $1 \le i \le n$.*

Proof. The extension is done as follows. We define $\tilde{v}_i$ for every edge v_i in the signed graph T, in a similar way as it was defined in [5, page 7]:

$$\tilde{v}_{a_v} = \begin{cases} v_{a_v}, & \text{for every edge } v \text{ signed by } a_v \text{ which does not} \\ & \text{touch the cycle} \\ u_{i+1}, & \text{for every edge } v_{a_v} = u_i \\ \bar{u}_{i+1}, & \text{for every edge } v_{a_v} = \bar{u}_i \\ u_{i+1} v_{a_v} u_{i+1}, & \text{for every edge } v \text{ signed by } a_v \text{ which does} \\ & \text{touch the cycle at vertex } x_i \text{ only} \\ u_{i+1} u_{j+1} v_{a_v} u_{j+1} u_{i+1}, & \text{for every edge } v \text{ signed by } a_v \text{ which does} \\ & \text{touch the cycle at vertices } x_i \text{ and } x_j \end{cases}$$

$$\gamma_t := \tilde{v}^{(s)}_{a_{v^{(s)}}} \cdots \tilde{v}^{(1)}_{a_{v^{(1)}}} \gamma_1 v^{(1)}{}_{a_{v^{(1)}}} \cdots v^{(s)}{}_{a_{v^{(s)}}}$$

and

$$\gamma_{\bar{t}} := \tilde{v}^{(s)}_{a_{v^{(s)}}} \cdots \tilde{v}^{(1)}_{a_{v^{(1)}}} \gamma_{\bar{1}} v^{(1)}{}_{a_{v^{(1)}}} \cdots v^{(s)}{}_{a_{v^{(s)}}}$$

where $v^{(1)}{}_{a_{v^{(1)}}}, \ldots, v^{(s)}{}_{a_{v^{(s)}}}$ is a connected path starting from the vertex 1 and ending at the vertex t in T_0. □

Proposition 21. *We extend the definition of δ_i also to every $1 \le i \le n$ in case there is a loop $w \notin T_0$. δ_i has already been defined for $\tilde{C}$-type cycle. Let $v^{(1)} \cdots v^{(s)}$ be a path from the vertex x_1 in the $\tilde{C}$-type cycle to a vertex $x_t \in C_Y(T)$. Then:*

$$\delta_t := {v^{(s)}}_{a_{v^{(s)}}} \cdots {v^{(1)}}_{a_{v^{(1)}}} \delta_1 {v^{(1)}}_{a_{v^{(1)}}} \cdots {v^{(s)}}_{a_{v^{(s)}}}$$

$$\delta_{\bar{t}} := {v^{(s)}}_{a_{v^{(s)}}} \cdots {v^{(1)}}_{a_{v^{(1)}}} \delta_{\bar{1}} {v^{(1)}}_{a_{v^{(1)}}} \cdots {v^{(s)}}_{a_{v^{(s)}}}$$

(Note: The definition of the extesion of δ is different from the definition of the extension of γ, and does not use the defined verices $\tilde{v}$).

Remark 22. *We notice that Propositions 17, 18 and 19 are holds for every $1 \leq i \leq n$. The proof is by looking at the elements γ_i and δ_i as elements of the defined group, and showing that the elements $\gamma_i^{-1}\gamma_j$ can be considered as elements of the extended periodic permutation group to a period of $2n+2$, where $\pi(j) = j + (2n+2)$ and $\pi(i) = i - (2n+2)$.*

Proof of Theorem 16:

Similarly as defined in [5], we define here $\theta : C_Y(T) \to A_{t,n} \rtimes G$, where t is the number of the cycles (every type) in the signed graph, and $G = B_n$ or D_n, depending on existence of loops in T.

For $u \in T$ we have $u = (ij)_a$ and

$$\theta(u) = \begin{cases} (ij)(\bar{i}\bar{j}), & \text{if } u \in T_0 \text{ and } a = 1 \\ (i\bar{j})(\bar{i}j), & \text{if } u \in T_0 \text{ and } a = -1 \\ (ij)(\bar{i}\bar{j})v_{ij}, & \text{if } v \notin T_0 \text{ and } a = 1 \\ (i\bar{j})(\bar{i}j)v_{\bar{i}j}, & \text{if } v \notin T_0 \text{ and } a = -1 \text{ and} \\ & v \text{ is not a loop in } \tilde{C} \text{ type cycle} \\ (i\bar{i})r_{v,i}, & \text{if } v \notin T_0 \text{ and } v \text{ is a loop} \end{cases}$$

We can show that θ is well-defined on $C_Y(T)$, i.e., the image of θ satisfies Relations (21), (22), (23) and (24).

- Relation (21) was treated in [5].
- (22) means that $\theta(uvu)$ commutes with $\theta(w\bar{v}w)$ for every $u, v, w \in T$ and $(uv)^3 = (vw)^3 = (uw)^2 = 1$. Now we treat the possible cases:
  (1) $u, v \in T_0$: $\theta(uvu) = (ik)(\bar{i}\bar{k})$,

(2) $u \notin T_0$, $v \in T_0$:

$\theta(u) = (ij)(\bar{i}\bar{j})u_{ij}$, $\theta(v) = (kj)(\bar{k}\bar{j})$, and $\theta(uvu) = (ij)(\bar{i}\bar{j})u_{ij}(kj)(\bar{k}\bar{l})(ij)(\bar{i}\bar{j})u_{ij} = (ik)(\bar{i}\bar{k})u_{ik}$,

(3) $u \in T_0$, $v \notin T_0$:

$\theta(u) = (ij)(\bar{i}\bar{j})$, $\theta(v) = (kj)(\bar{k}\bar{j})v_{kj}$, and $\theta(uvu) = (ij)(\bar{i}\bar{j})(kj)(\bar{k}\bar{j})v_{kj}(ij)(\bar{i}\bar{j}) = (ik)(\bar{i}\bar{k})v_{ki}$,

(4) $u, v \notin T_0$:

$\theta(u) = (ij)(\bar{i}\bar{j})u_{ij}$, $\theta(v) = (kj)(\bar{k}\bar{j})v_{kj}$, and $\theta(uvu) = (ik)(\bar{i}\bar{k})u_{jk}v_{ki}u_{ij}$ (see proof in [5]).

Similarly, $\theta(w\bar{v}w)$ is one of the followings:

(1) $w, \bar{v} \in T_0$: $\theta(w\bar{v}w) = (l\bar{j})(\bar{l}j)$,

(2) $w \notin T_0$, $\bar{v} \in T_0$: $\theta(w\bar{v}w) = (l\bar{j})(\bar{l}j)w_{l\bar{j}}$,

(3) $w \in T_0$, $\bar{v} \notin T_0$: $\theta(w\bar{v}w) = (l\bar{j})(\bar{l}j)\bar{v}_{\bar{j}l}$,

(4) $w, \bar{v} \notin T_0$: $\theta(w\bar{v}w) = (l\bar{j})(\bar{l}j)w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk}$.

Since i, k, l and $\bar{j}$ are distinct, each one of the elements $(ik)(\bar{i}\bar{k}), (ik)(\bar{i}\bar{k})u_{ik}, (ik)(\bar{i}\bar{k})v_{ki}$ commutes with each one of the elements $(l\bar{j})(\bar{l}j), (l\bar{j})(\bar{l}j)w_{l\bar{j}}, (l\bar{j})(\bar{l}j)\bar{v}_{\bar{j}l}$.

It remains to show that each one of the elements $(ik)(\bar{i}\bar{k}), (ik)(\bar{i}\bar{k})u_{ik}, (ik)(\bar{i}\bar{k})v_{ki}$, and $(ik)(\bar{i}\bar{k})u_{jk}v_{ki}u_{ij}$ commutes with $(l\bar{j})(\bar{l}j)w_{k\bar{j}}\bar{v}_{\bar{j}l}(j\bar{l})w_{lk}$. We start with (for distinct i, j, k, l):

$$(l\bar{j})(\bar{l}j)w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk}(ik)(\bar{i}\bar{k}) = (l\bar{j})(\bar{l}j)(ik)(\bar{i}\bar{k})w_{i\bar{j}}\bar{v}_{\bar{j}l}w_{li} = (ik)(\bar{i}\bar{k})(l\bar{j})(\bar{l}j)w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk}, \text{by (35)}.$$

Now we prove:

$$(l\bar{j})(\bar{l}j)w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk}(ik)(\bar{i}\bar{k})u_{ik} = (ik)(\bar{i}\bar{k})(l\bar{j})(\bar{l}j)w_{i\bar{j}}\bar{v}_{\bar{j}l}w_{li}u_{ik}$$

$$\overset{(\overline{\overline{35}})}{=} (ik)(\bar{i}\bar{k})(l\bar{j})(\bar{l}j)u_{ik}w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk} = (ik)(\bar{i}\bar{k})u_{ik}(l\bar{j})(\bar{l}j)w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk}.$$

Similarly,

$$(l\bar{j})(\bar{l}j)w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk}(ik)(\bar{i}\bar{k})v_{ki} = (ik)(\bar{i}\bar{k})v_{ki}(l\bar{j})(\bar{l}j)w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk}.$$

Now we show that if $n \geq 6$, then $(ik)(\bar{i}\bar{k})u_{jk}v_{ki}u_{ij}$ commutes with $(l\bar{j})(\bar{l}j)w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk}$. Since $n \geq 6$, there exist p and q such that $i, j, \bar{j}, k, l, p$ and q are distinct and by (36),

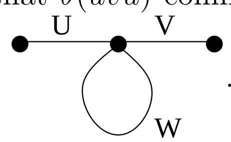
we have: $u_{jk}v_{ki}u_{ij} = u_{pk}v_{ki}u_{ip}$ and $w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk} = w_{q\bar{j}}\bar{v}_{\bar{j}l}w_{lq}$. Hence:

$$(ik)(\bar{i}\bar{k})u_{jk}v_{ki}u_{ij}(l\bar{j})(\bar{l}j)w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk} \underset{(35)}{=} (ik)(\bar{i}\bar{k})u_{pk}v_{ki}u_{ip}(l\bar{j})(\bar{l}j)w_{q\bar{j}}\bar{v}_{\bar{j}l}w_{lq}$$

$$= (ik)(\bar{i}\bar{k})(l\bar{j})(\bar{l}j)u_{pk}v_{ki}u_{ip}w_{q\bar{j}}\bar{v}_{\bar{j}l}w_{lq} \underset{(36)}{=} (ik)(\bar{i}\bar{k})(l\bar{j})(\bar{l}j)w_{q\bar{j}}\bar{v}_{\bar{j}l}w_{lq}u_{pk}v_{ki}u_{ip}$$

$$= (l\bar{j})(\bar{l}j)w_{q\bar{j}}\bar{v}_{\bar{j}l}w_{lq}(ik)(\bar{i}\bar{k})u_{pk}v_{ki}u_{ip} = (l\bar{j})(\bar{l}j)w_{q\bar{j}}\bar{v}_{\bar{j}l}w_{lq}(ik)(\bar{i}\bar{k})u_{pk}v_{ki}u_{ip}$$

$$\underset{(35)}{=} (l\bar{j})(\bar{l}j)w_{k\bar{j}}\bar{v}_{\bar{j}l}w_{lk}(ik)(\bar{i}\bar{k})u_{jk}v_{ki}u_{ij}.$$

- (23) means that $\theta(uvu)$ commutes with $\theta(w)$ for $u, v, w \in T$ and $(uv)^3 = (uw)^4 = (vw)^4 = 1$, 

.
  The proof is the same one for the 'fork' relation in [5, P. 20].
- (24) means that $(\theta(u) \cdot \theta(wvw))^3 = 1$ for $u, v, w \in T$ and $(uv)^3 = (uw)^4 = (vw)^4 = 1$.

Now we classify the possible cases for $\theta(wvw)$ and $\theta(u)$. We start with $\theta(wvw)$:

(1) $v, w \in T_0$:
$\theta(v) = (kj)(\bar{k}\bar{j})$, $\theta(w) = (\bar{j}j)$, and $\theta(wvw) = (\bar{j}k)(j\bar{k})$.

(2) $v \notin T_0$, $w \in T_0$:
$\theta(v) = (kj)(\bar{k}\bar{j})v_{kj}$, $\theta(w) = (\bar{j}j)$, and $\theta(wvw) = (\bar{j}k)(j\bar{k})v_{k\bar{j}}$.

(3) $v \in T_0$, $w \notin T_0$:
$\theta(v) = (kj)(\bar{k}\bar{j})$, $\theta(w) = (\bar{j}j)r_{w,\bar{j}}$, and $\theta(wvw) = (\bar{j}k)(j\bar{k})r_{w,\bar{k}}r_{w,\bar{j}}$.

(4) $v, w \notin T_0$:
$\theta(v) = (kj)(\bar{k}\bar{j})v_{kj}$, $\theta(w) = (\bar{j}j)r_{w,\bar{j}}$, and $\theta(wvw) = (\bar{j}k)(j\bar{k})r_{w,\bar{k}}v_{k\bar{j}}r_{w,\bar{j}}$.

And here we give the following forms of $\theta(u)$:

(a) $u \in T_0$: $\theta(u) = (ij)(\bar{i}\bar{j})$,

(b) $u \notin T_0$: $\theta(u) = (ij)(\bar{i}\bar{j})u_{ij}$.

In the case (1) and (a) we have:

$$(\theta(u) \cdot \theta(wvw))^3 = ((ij)(\bar{i}\bar{j}) \cdot (\bar{j}k)(j\bar{k}))^3 = [(i\bar{k}j)(\bar{i}j\bar{k})]^3 = 1.$$

In the case (2) and (a) we have:

$$(\theta(u) \cdot \theta(wvw))^3 = (ij)(\bar{i}\bar{j})(\bar{j}k)(j\bar{k})v_{k\bar{j}}(ij)(\bar{i}\bar{j})(\bar{j}k)(j\bar{k})v_{k\bar{j}}(ij)(\bar{i}\bar{j})(\bar{j}k)(j\bar{k})v_{k\bar{j}}$$

$$= (i\bar{k}j)(\bar{i}j\bar{k})v_{k\bar{j}}(i\bar{k}j)(\bar{i}k\bar{j})v_{k\bar{j}}(i\bar{k}j)(\bar{i}k\bar{j})v_{k\bar{j}} = v_{\bar{i}k}v_{\bar{j}\bar{i}}v_{k\bar{j}} = 1.$$

In the case (3) and (a) we have:

$$(\theta(u)\cdot\theta(wvw))^3 = (ij)(\bar{i}\bar{j})(\bar{j}k)(j\bar{k})r_{w,\bar{k}}r_{w,\bar{j}}(ij)(\bar{i}\bar{j})(\bar{j}k)(j\bar{k})r_{w,\bar{k}}r_{w,\bar{j}}(ij)(\bar{i}\bar{j})(\bar{j}k)(j\bar{k})r_{w,\bar{k}}r_{w,\bar{j}}$$
$$= (i\bar{k}j)(\bar{i}j\bar{k})r_{w,\bar{k}}r_{w,\bar{j}}(i\bar{k}j)(\bar{i}j\bar{k})r_{w,\bar{k}}r_{w,\bar{j}}(i\bar{k}j)(\bar{i}k\bar{j})r_{w,\bar{k}}r_{w,\bar{j}} = r_{w,i}r_{w,k}r_{w,j}r_{w,\bar{i}}r_{w,\bar{k}}r_{w,\bar{j}} = 1.$$

In the case (4) and (a) we have (by Relation (31)):

$$(\theta(u)\cdot\theta(wvw))^3 = (i\bar{k}j)(\bar{i}j\bar{k})r_{w,\bar{k}}v_{k\bar{j}}r_{w,\bar{j}}(i\bar{k}j)(\bar{i}k\bar{j})r_{w,\bar{k}}v_{k\bar{j}}r_{w,\bar{j}}(i\bar{k}j)(\bar{i}k\bar{j})r_{w,\bar{k}}v_{k\bar{j}}r_{w,\bar{j}}$$
$$= r_{w,i}v_{\bar{i}k}r_{w,k}r_{w,j}v_{\bar{j}\bar{i}}r_{w,\bar{i}}r_{w,\bar{k}}v_{k\bar{j}}r_{w,\bar{j}} = 1.$$

In the case (1) and (b) we have (as in the case of (2) and (a)):

$$(\theta(wvw)\cdot\theta(u))^3 = ((k\bar{j})(\bar{k}j)(ij)(\bar{i}\bar{j})u_{ij})^3 = ((ij\bar{k})(\bar{i}\bar{j}k)u_{ij})^3 = 1.$$

In the case (2) and (b) we have:

$$(\theta(u)\cdot\theta(wvw))^3 = ((ij)(\bar{i}\bar{j})u_{ij}(j\bar{k})(\bar{j}k)v_{k\bar{j}})^3 = ((i\bar{k}j)(\bar{i}k\bar{j})u_{i\bar{k}}v_{k\bar{j}})^3 = u_{ji}v_{\bar{i}k}u_{\bar{k}j}v_{\bar{j}\bar{i}}u_{i\bar{k}}v_{k\bar{j}} = 1.$$

In the case (3) and (b) we have:

$$(\theta(u)\cdot\theta(wvw))^3 = ((ij)(\bar{i}\bar{j})u_{ij}(j\bar{k})(\bar{j}k)r_{w,\bar{k}}r_{w,\bar{j}})^3 = ((i\bar{k}j)(\bar{i}k\bar{j})u_{i\bar{k}}r_{w,\bar{k}}r_{w,\bar{j}})^3$$
$$= u_{ji}r_{w,i}r_{w,k}u_{\bar{k}j}r_{w,j}r_{w,\bar{i}}u_{i\bar{k}}r_{w,\bar{k}}r_{w,\bar{j}} = 1.$$

In the case (4) and (b) we have:

$$(\theta(u)\cdot\theta(wvw))^3 = ((ij)(\bar{i}\bar{j})u_{ij}(\bar{j}k)(j\bar{k})r_{w,\bar{k}}v_{k\bar{j}}r_{w,\bar{j}})^3 = ((i\bar{k}j)(\bar{i}k\bar{j})u_{i\bar{k}}r_{w,\bar{k}}v_{k\bar{j}}r_{w,\bar{j}})^3$$
$$= u_{ji}r_{w,i}v_{\bar{i}k}r_{w,k}u_{\bar{k}j}r_{w,j}v_{\bar{j}\bar{i}}r_{w,\bar{i}}u_{i\bar{k}}r_{w,\bar{k}}v_{kj}r_{w,\bar{j}} = 1.$$

We conclude that the Relations (21), (22), (23) and (24) are satisfied for $\theta(C_Y(T))$. Hence θ is well defined on $C_Y(T)$.

This proves $\theta : C_Y(T) \to A_{t,n} \rtimes G$ is a homomorphism. $G = Im(\phi(C(T))$, where ϕ is the natural map from $C(T)$ into B_n which was defined in Lemma 5. By the same Lemma $G = B_n$ in case T does contain a loop, or $G = D_n$ in case T does not contain a loop but does conatain an anti-cycle.

Now we define $\tau : A_{t,n} \rtimes G \to C_Y(T)$ as it was defined in [5]:
$\tau(v) = v$ if $v \in B_n$ or $v \in D_n$, $\tau(x_{ij}) = {\gamma_j}^{-1}\gamma_i$ and $\tau(r_{x,i}) = \delta_i$.

We need to check Relations (25), (26), (27) and (29) for $\tau(x_{i,j})$. Relation (25) holds trivially since $\gamma_i^{-1}\gamma_i = 1$. Relation (26) holds, since $(\gamma_i^{-1}\gamma_j)^{-1} = \gamma_j^{-1}\gamma_i$, and Relation (27) and (29) hold by Proposition 18.

Hence, τ is well defined, and τ is the inverse map of θ. There are five options:

1) $\theta\tau(x_{ij}) = \theta(\gamma_j^{-1}\gamma_i)$. Then the proof is exactly the same as in [5].

2) $\theta\tau(x_{\bar{i}\bar{j}}) = \theta(\gamma_{\bar{j}}^{-1}\gamma_{\bar{i}})$. By Proposition 17: $\gamma_{\bar{j}}^{-1}\gamma_{\bar{i}} = \gamma_i^{-1}\gamma_j$, and $\tau(\gamma_{\bar{j}}^{-1}\gamma_{\bar{i}}) = x_{ij} = x_{\bar{j}\bar{i}}$, by Relation (30).

3) $\theta\tau(x_{\bar{i}j}) = \theta(\gamma_j^{-1}\gamma_{\bar{i}}) =$

$= \theta(u_j u_{j-1} \cdots u_1 u_m x_{0m} u_{m-1} \cdots u_{j+2} u_{i+1} u_i \bar{u}_i u_{i+1} \cdots u_m x_{0m} u_1 \cdots u_{i-2} \bar{u}_{i-1} u_i) =$

$= x_{\bar{i}j}$.

4) $\theta\tau(x_{i\bar{j}}) = \theta(\gamma_{\bar{j}}^{-1}\gamma_i) = \theta(\gamma_i^{-1}\gamma_{\bar{j}})^{-1} = x_{\bar{j}i}^{-1} = x_{i\bar{j}}$ by Relation (26).

5) $\theta\tau(r_{x,i}) = \theta(\delta_i) = \theta(u_i u_{i-1} \cdots u_1 v u_1 \cdots u_{m-1} w u_{m-1} \cdots u_{i+1}) =$

$= u_i u_{i-1} \cdots u_1 v u_1 \cdots u_{m-1} w r_{x,m-1} u_{m-1} \cdots u_{i+1} = r_{x,i}$.

Hence, in every case $\theta\tau$ is the identity, then τ is the inverse map to θ. $\square$

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